Readers' Forum

Brief discussions of previous investigations in the aerospace sciences and technical comments on papers published in the AIAA Journal *are presented in this special department. Entries must be restricted to a maximum of 1000 words, or the equivalent of one* Journal *page including formulas and figures. A discussion will be published as quickly as possible after receipt of the manuscript. Neither the AIAA nor its editors are responsible for the opinions expressed by the correspondents. Authors will be invited to reply promptly.*

Comment on "Numerical Evaluation of Principal Value Integrals by Gauss-Laguerre Quadrature"

William Squire*
West Virginia University, Morgantown, West Virginia

THE evaluation of principal value integrals arises in several disciplines. Unfortunately, communication between workers in different fields has been poor. I thought I had a reasonably complete set of references, but Endo[1] has brought a new group to my attention.

Most of the work has been done for integrals over a finite range, but the methods are easily adapted to a semi-infinite range. A critical survey by van der Sluis and Zweerus[2] describes the three principal approaches: subtraction of the singualrity, folding, and symmetrization.

Subtraction of the Singularity

If we write

$$I(a)=\int_0^\infty F_a(x)\,\mathrm{d}x=\int_0^\infty [F_a(x)-G_a(x)]\mathrm{d}x+\int_0^\infty G_a(x)\,\mathrm{d}x \tag{1}$$

where G_a† is a function with a matching singularity at a and for which the principal value integral can be evaluated explicitly, then the problem reduces to numerical evaluation of an ordinary integral. Endo takes advantage of the special form of the integrand in the problem he is interested in to use Gauss-Laguerre quadrature. A promising alternative would be the quadrature rule

$$\int_0^\infty f\mathrm{d}x \cong \ell n r \sum_{J=0}^{N} x_0 r^j f(x_0 r^j) \tag{2}$$

using nodes in a geometric progression. This would apply to a wider range of integrands and would not, in many cases, require more nodes.

Folding

If we split the integral at the singularity so that

$$I(a)=\int_0^a F_a(x)\,\mathrm{d}x+\int_0^\infty F_a(x)\,\mathrm{d}x \tag{3}$$

we can transform the two integrals to the same range to obtain:

$$I(a)=a\int_0^1 F_1(ax)+x^{-2}F_1\left(\frac{a}{x}\right)\mathrm{d}x \tag{4a}$$

or

$$I(a)=a\int_1^\infty F_1(ax)+x^{-2}F_1\left(\frac{a}{x}\right)\mathrm{d}x \tag{4b}$$

In both cases the integrand becomes indeterminate at $x=1$ but can be evaluated by a limiting process such as L'Hopital's rule. However, this can be avoided by using Eq. (2) for Eq. (4b) and an open quadrature rule such as Gauss quadrature for Eq. (4a).

Symmetrization

It has been known for about twenty years that principal value integrals can be evaluated accurately by using a quadrature rule which is symmetric with respect to the singularity. The Monocella approach referred to by Endo uses the trapezoid or midpoint rule and places the singularity midway between two nodes. The drawback for the semi-infinite case is the large number of nodes required.

An alternative symmetrization involves splitting the integral at $2a$ so that

$$I(a)=\int_0^{2a} G_a(x)\,\mathrm{d}x+\int_{2a}^\infty G(x)\,\mathrm{d}x \tag{5}$$

The first integral could be evaluated by an even-order Gauss rule which would be symmetric about the midpoint. The second integral could be evaluated by Eq. (2).

A report comparing the results obtained by these different methods will be submitted for publication shortly.

References

[1] Endo, H., "Numerical Evaluation of Principal Value Integral by Gauss-Laguerre Quadrature," *AIAA Journal*, Vol. 21, Jan. 1983, pp. 149-151.

[2] van der Sluis, A. and Zweerus, M., "An Appraisal of Some Methods for Computing Cauchy Principal Values of Integrals," *Numerische Integration*, ISNM, Vol. 95, 1979, pp. 264-287.

[3] Squire, W., "An Efficient Iterative Method for Numerical Evaluation of Integrals Over a Semi-Infinite Range," *International Journal of Numerical Methods in Engineering*, Vol. 10, No. 2, 1976, pp. 478-484.

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*Professor, Mechanical and Aerospace Engineering. Associate Fellow AIAA.

†The subscript indicates the location of the singularity.

Reply by Author to W. Squire

Hideichi Endo*
Hideichi Zosen Corporation, Osaka, Japan

THANK you very much for your interest in my paper and for introducing an interesting Comment on different approaches to solving principal value integrals.

As far as the first approach (subtraction of the singularity) is concerned, the problem still remains in the second integral, which is a principal value integral. In my case, I happened to find the exponential integral that is suitable to my problem; hence, I could use the first technique. However, I do not think that the first technique is always adaptable to all kinds of principal value integrals. In that case we have to use another approach. I think your coming paper will be very interesting and helpful, and I am looking forward to reading it.

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*Researcher, Technical Research Institute.